Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number	Candidate Number

Time 1 hour 30 minutes

Paper reference **9FM0/3A**

Further Mathematics

Advanced

PAPER 3A: Further Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- Good luck with your examination.

Turn over

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Pearson

1. The ellipse E has equation

$$\frac{x^2}{36} + \frac{y^2}{20} = 1$$

Find

(a) the coordinates of the foci of E, **(3)**

(b) the equations of the directrices of E. **(2)**

rewrite: $\frac{x^2}{(6)^2} + \frac{y^2}{(2\sqrt{5})^2} = 1$

$a = 6$ $b = 2\sqrt{5}$

a. foci : $(\pm ae, 0)$

6 must work out

$b^2 = a^2(1-e^2)$

$20 = 36(1-e^2)$

$\frac{20}{36} = 1 - e^2$

$e^2 = 1 - \frac{20}{36} = \frac{16}{36}$

$e = \pm\sqrt{\frac{16}{36}} = \pm 4/6 = \pm 2/3$

$0 < e < 1$

$\therefore$ $e = +2/3$

foci : $\left(\pm (6)(\tfrac{2}{3}), 0\right)$

foci : $(\pm 4, 0)$

table in F.B.

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e < 1$ $b^2 = a^2(1 - e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2 - 1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x = \pm\frac{a}{e}$	$x = -a$	$x = \pm\frac{a}{e}$	$x + y = \pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

b. directrices : $x = \pm\frac{a}{e}$

$x = \pm\frac{(6)}{(\frac{2}{3})} = \frac{18}{2} = 9$

$x = \pm 9$

2. (i) Use the substitution $t = \tan\frac{x}{2}$ to prove the identity

$$\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} \equiv \sec x + \tan x \qquad x \neq \frac{n\pi}{2} \qquad n \in \mathbb{Z}$$

(5)

(ii) Use the substitution $t = \tan\frac{\theta}{2}$ to determine the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{5}{4 + 2\cos\theta}\, d\theta$$

giving your answer in simplest form.

(5)

i. if $t = \tan\left(\frac{x}{2}\right)$, $\sin(x) = \frac{2t}{1+t^2}$

$\cos(x) = \frac{1-t^2}{1+t^2}$

proof @ end of Q

Start from LHS:

$$\frac{\left(\frac{2t}{1+t^2}\right) - \left(\frac{1-t^2}{1+t^2}\right) + 1}{\left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right) - 1}$$

$$= \frac{\frac{2t - (1-t^2) + (1+t^2)}{\cancel{1+t^2}}}{\frac{2t + (1-t^2) - (1+t^2)}{\cancel{1+t^2}}}$$

$$= \frac{2t - (1-t^2) + (1+t^2)}{2t + (1-t^2) - (1+t^2)}$$

$$= \frac{2t \cancel{-1} + t^2 \cancel{+1} + t^2}{2t \cancel{+1} - t^2 \cancel{-1} - t^2}$$

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Question 2 continued

$$= \frac{2t^2 + 2t}{-2t^2 + 2t}$$

$$= \frac{\cancel{2t}(t+1)}{\cancel{2t}(t-1)}$$

$$= \frac{t+1}{t-1}$$

multiply by $t+1$ on both sides as this will lead to t^2-1 on denominator which can be split.

$$\frac{t+1}{t-1} \times \frac{t+1}{t+1} = \frac{t^2+2t+1}{t^2-1}$$

difference of 2 squares

$\sec x = \frac{1}{\cos x} = \frac{1+t^2}{1-t^2}$

$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{2t}{\cancel{1-t^2}}}{\frac{1+t^2}{\cancel{1-t^2}}} = \frac{2t}{1+t^2}$

$$\frac{t^2+2t+1}{t^2-1} = \frac{t^2+1}{t^2-1} + \frac{2t}{1-t^2}$$

$$= \sec x + \tan x \text{ (Ans)}$$

ii. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{5}{4+2\cos\theta}\, d\theta$

$t = \tan\left(\frac{\theta}{2}\right) \Rightarrow \cos(\theta) = \frac{1-t^2}{1+t^2}$

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)$

$\frac{dt}{d\theta} = \frac{1}{2}\left[1+\tan^2\left(\frac{\theta}{2}\right)\right] = \frac{1+t^2}{2}$

$d\theta = \frac{2}{1+t^2}\, dt$

P 6 6 7 9 8 A 0 5 3 2

Turn over

Question 2 continued

Change limits:

$\theta = 0$ $\qquad \theta = \frac{\pi}{2}$

$t = \tan\left(\frac{0}{2}\right) = 0$ $\qquad t = \tan\left(\frac{\pi/2}{2}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$$\int_0^1 \frac{5}{4 + 2\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}\,dt$$

$$\int_0^1 \frac{10}{\frac{4+4t^2+2-2t^2}{\cancel{1+t^2}}} \times \frac{dt}{\cancel{1+t^2}}$$

$$\int_0^1 \frac{10}{2t^2+6}\,dt$$

$$\int_0^1 \frac{\cancel{10}\,5}{\cancel{2}(t^2+3)}\,dt$$

$$\int_0^1 \frac{5}{t^2+3}\,dt$$

$$= 5\int_0^1 \frac{1}{t^2+3}\,dt$$

in form as: $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$ $\qquad a > 0$

(in formulae booklet)

$a^2 = 3$

$a = +\sqrt{3}$

P 6 6 7 9 8 A 0 6 3 2

Question 2 continued

$$5\left[\frac{1}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right)\right]_0^1$$

$$5\left[\left(\frac{1}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right)\right)-\cancel{\left(\frac{1}{\sqrt{3}}\arctan\left(\frac{0}{\sqrt{3}}\right)\right)}\right]$$

$$\downarrow 0$$

$$= 5\left[\left(\frac{1}{\sqrt{3}}\right)\left(\frac{\pi}{6}\right)\right]$$

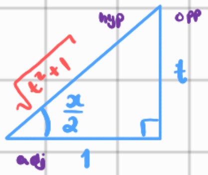
$$= \frac{5\pi}{6\sqrt{3}} \times \frac{6\sqrt{3}}{6\sqrt{3}}$$ rationalise surd denominator

$$= \frac{\cancel{30}^{5}\pi\sqrt{3}}{\cancel{108}_{18}}$$

$$= \frac{5\pi\sqrt{3}}{18}$$

(Total for Question 2 is 10 marks)

Deriving the t-formulae:



1) draw a right-angled triangle and label angle and sides which you know.

* you are allowed to memorise the t-formulae for sin(x), cos(x), tan(x) and do not have to derive it in the exam unless specifically asked.

2) work out hypotenuse in terms of t, (pythagoras)

$$\sqrt{(t)^2+(1)^2}$$
$$= \sqrt{t^2+1}$$

3) label each side of triangle opposite, adjacent, hypotenuse

4) write out $\sin\left(\frac{x}{2}\right), \cos\left(\frac{x}{2}\right), \tan\left(\frac{x}{2}\right)$ in terms of t.

$$\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{t^2+1}} \qquad \sin = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{t^2+1}} \qquad \cos = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{t}{1} = t \qquad \tan = \frac{\text{opposite}}{\text{adjacent}} \Big/ \frac{\sin}{\cos}$$

5) Now use double-angle formulae and write in terms of t:

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$
$$\cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$\sin(x) = 2\left(\frac{t}{\sqrt{t^2+1}}\right)\left(\frac{1}{\sqrt{t^2+1}}\right) \qquad \cos(x) = \left(\frac{1}{\sqrt{t^2+1}}\right)^2 - \left(\frac{t}{\sqrt{t^2+1}}\right)^2$$

$$\sin(x) = \frac{2t}{t^2+1} \qquad \cos(x) = \frac{1-t^2}{1+t^2}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\frac{2t}{\cancel{1+t^2}}}{\frac{1-t^2}{\cancel{1+t^2}}} = \frac{2t}{1-t^2}$$

$$\tan(x) = \frac{2t}{1-t^2}$$

3.

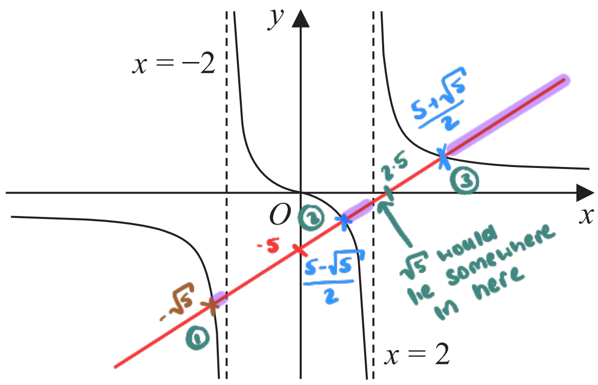


Figure 1

Figure 1 shows a sketch of the curve with equation $y = f(x)$ where

$$f(x) = \frac{x}{|x| - 2}$$

Use algebra to determine the values of x for which

$$2x - 5 > \frac{x}{|x| - 2}$$

(8)

Sketch $y = 2x-5$ on graph roughly

: shows part of graph where $2x-5 > \frac{x}{|x|-2}$

equate both eq^n's normally and find intersection points.
↳ must do for $\frac{x}{x-2}$ and $\frac{x}{-x-2}$
↑ negated

$2x-5 = \frac{x}{x-2}$

$(2x-5)(x-2) = x$
$2x^2 - 9x + 10 = x$
$2x^2 - 10x + 10 = 0$

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Question 3 continued

$x^2-5x+5=0$

$x = \dfrac{-(-5) \pm \sqrt{(-5)^2-(4)(1)(5)}}{2(1)}$

$x = \dfrac{5 \pm \sqrt{5}}{2}$

plot on graph

$2x-5 = \dfrac{x}{-x-2}$

$(2x-5) = \dfrac{x}{-(x+2)}$

$(2x-5)(x+2) = -x$

$2x^2-5x+4x-10 = -x$

$2x^2-10=0$

$x^2=5$

$x = \pm\sqrt{5}$

plot on graph

③ The 3rd point of intersection may be either $\sqrt{5}$ or $\dfrac{5+\sqrt{5}}{2}$

we pick ③ as $\dfrac{5+\sqrt{5}}{2}$ because

$\sqrt{5} \approx 2.23\ldots$

The line $y=2x-5$ intersects x-axis @ $(2.5, 0)$

$\therefore$ $\sqrt{5}$ would be between asymptote and x-axis.

$-\sqrt{5} < x < -2$ or $\dfrac{5-\sqrt{5}}{2} < x < 2$ or $x > \dfrac{5+\sqrt{5}}{2}$ //

4.

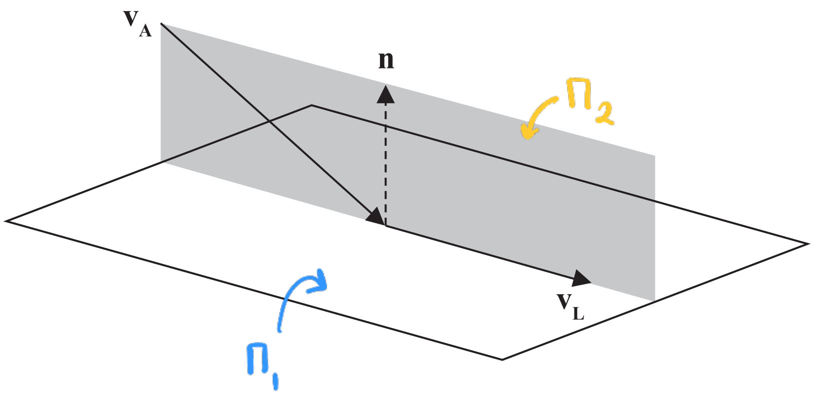

Figure 2

A small aircraft is landing in a field.

In a model for the landing the aircraft travels in different straight lines before and after it lands, as shown in Figure 2.

The vector $\mathbf{v}_A$ is in the direction of travel of the aircraft as it approaches the field.

The vector $\mathbf{v}_L$ is in the direction of travel of the aircraft after it lands.

With respect to a fixed origin, the field is modelled as the plane with equation

$$x - 2y + 25z = 0$$

$\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 25 \end{pmatrix} = 0$

and

$$\mathbf{v}_A = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

(a) Write down a vector $\mathbf{n}$ that is a normal vector to the field.

(1)

(b) Show that $\mathbf{n} \times \mathbf{v}_A = \lambda \begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix}$, where λ is a constant to be determined.

(2)

When the aircraft lands it remains in contact with the field and travels in the direction $\mathbf{v}_L$

The vector $\mathbf{v}_L$ is in the same plane as both $\mathbf{v}_A$ and $\mathbf{n}$ as shown in Figure 2.

(c) Determine a vector which has the same direction as $\mathbf{v}_L$

(3)

(d) State a limitation of the model.

(1)

a. $\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ 25 \end{pmatrix}$

P 6 6 7 9 8 A 0 1 2 3 2

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Question 4 continued

b. $n \times V_a = \lambda \begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ -2 \\ 25 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = \begin{array}{ccc} i & j & k \\ \hline 1 & -2 & 25 \\ 3 & -2 & -1 \end{array}$$

$$i \begin{vmatrix} -2 & 25 \\ -2 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & 25 \\ 3 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & -2 \\ 3 & -2 \end{vmatrix}$$

$$i\left[(-2)(-1) - (25)(-2)\right] - j\left[(1)(-1) - (3)(25)\right] + k\left[(1)(-2) - (-2)(3)\right]$$

$52i + 76j + 4k$

$$\begin{pmatrix} 52 \\ 76 \\ 4 \end{pmatrix} = \lambda \begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix}$$

$\lambda = 4$ //

c. we want to find Π_2

normal vector : $\begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix}$

Π eq^n formulae

general point in plane

$r \cdot n = a \cdot n$ $\qquad r \cdot n = d$

normal vector

$$r \cdot \begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix}}_{\text{dot product}}$$

dot product : $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf$

$$r \cdot \begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix} = (3)(13) + (-2)(19) + (-1)(1)$$

P 6 6 7 9 8 A 0 1 3 3 2

Question 4 continued

$r \cdot \begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix} = 0$

$\Pi_2 : \ 13x + 19y + z = 0$

solve Π_1 and Π_2 simultaneously

$x - 2y + 25z = 0$
$13x + 19y + z = 0$

set x, y or z to be any non-zero value.

e.g let $z = 1$

$x - 2y + 25 = 0$
$13x + 19y + 1 = 0$

$x - 2y = -25$ ①
$13x + 19y = -1$ ②

① × 13

$13x - 26y = -325$
$13x + 19y = -1$ ⊖

$-45y = -324$

$y = \frac{-324}{-45} = \frac{36}{5}$

$x = 2\left(\frac{36}{5}\right) - 25 = -\frac{53}{5}$

$\left(-\frac{53}{5}, \frac{36}{5}, 1\right)$ or any multiple //

a. planes might not be flat //

P 6 6 7 9 8 A 0 1 4 3 2

5. The parabola C has equation

$$y^2 = 32x$$

and the hyperbola H has equation

$$\frac{x^2}{36} - \frac{y^2}{9} = 1$$

(a) Write down the equations of the asymptotes of H.

(1)

The line l_1 is normal to C and parallel to the asymptote of C with positive gradient.

The line l_2 is normal to C and parallel to the asymptote of C with negative gradient.

(b) Determine

(i) an equation for l_1

(ii) an equation for l_2

(4)

The lines l_1 and l_2 meet H at the points P and Q respectively.

(c) Find the area of the triangle OPQ, where O is the origin.

(4)

a. rewrite as: $\frac{x^2}{(6)^2} - \frac{y^2}{(3)^2} = 1$

$a = 6$ $b = 3$

asymptotes: $\frac{x}{a} = \pm \frac{y}{b}$

$y = \pm \frac{b}{a} x$

$y = \pm \frac{3}{6} x = \pm \frac{1}{2} x$

$y = \pm \frac{1}{2} x$

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e < 1$ $b^2 = a^2(1 - e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2 - 1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x = \pm\frac{a}{e}$	$x = -a$	$x = \pm\frac{a}{e}$	$x + y = \pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

Question 5 continued

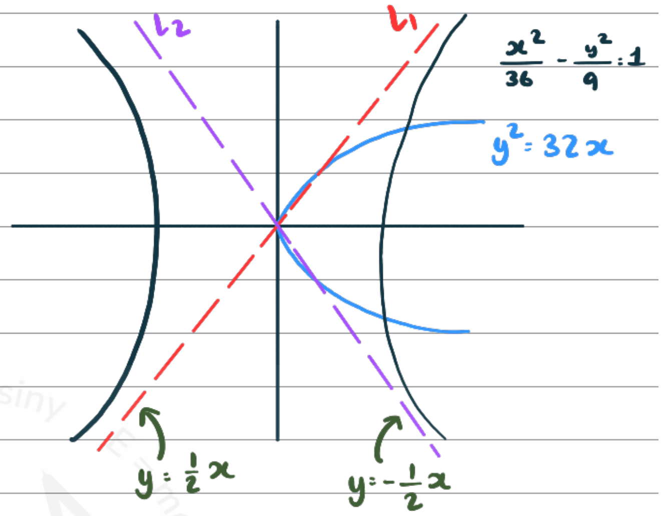
bi. $y^2 = 32x$

$2y\left(\frac{dy}{dx}\right) = 32$

$\frac{dy}{dx} = \frac{32}{2y} = \frac{16}{y}$

$M_{tangent} = \frac{16}{y}$

$M_{tangent} \times M_{normal} = -1$



$M_{normal} = -\frac{y}{16}$

$-\frac{y}{16} = \frac{1}{2}$

$y = \frac{1}{2} \times -16 = -8$

$y = -8$

sub into parabola eq^n to find x-coord.

$(-8)^2 = 32x$ $\quad$ $y^2 = 32x$

$64 = 32x$

$x = 2$

$(2, -8)$

find line eq^n in form $y - y_1 = m(x - x_1)$

$(x_1, y_1) = (2, -8)$

$m = \frac{1}{2}$

$y - (-8) = \frac{1}{2}(x - (2))$

$y + 8 = \frac{1}{2}(x - 2)$

$y + 8 = \frac{1}{2}x - 1$

$L_1: \; y = \frac{1}{2}x - 9$ //

ii. $M_{normal} = -\frac{y}{16}$

$-\frac{y}{16} = -\frac{1}{2}$

P 6 6 7 9 8 A 0 1 7 3 2

Question 5 continued

$y = -\frac{1}{2}x - 16$

$y = 8$ sub into parabola eqⁿ to find x-coord.

$(8)^2 = 32x$ $\quad y^2 = 32x$

$64 = 32x$

$x = 2$

$(2, 8)$

find line eqⁿ in form

$y - y_1 = m(x - x_1)$

$(x_1, y_1) = (2, 8)$

$m = 1/2$

$y - (8) = -\frac{1}{2}(x - (2))$

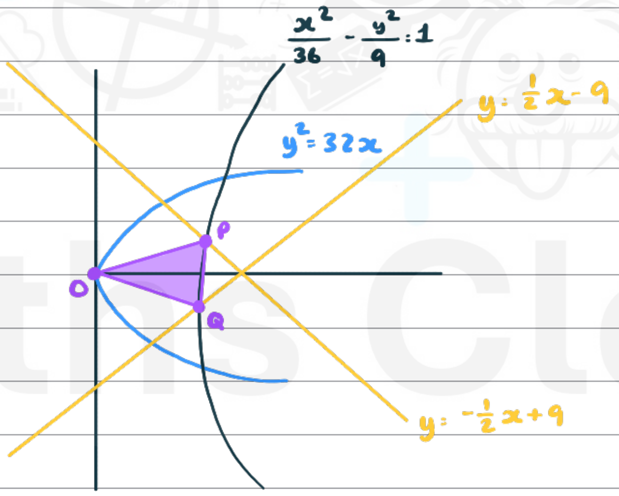
$y - 8 = -\frac{1}{2}(x - 2)$

$y - 8 = -\frac{1}{2}x + 1$

$y = -\frac{1}{2}x + 9$

$L_2 : y = -\frac{1}{2}x + 9$ //

c. find P and Q by solving $\frac{x^2}{36} - \frac{y^2}{9} = 1$ and each $y = \frac{1}{2}x - 9$ and $y = -\frac{1}{2}x + 9$



$\frac{x^2}{36} - \frac{(\frac{1}{2}x - 9)^2}{9} = 1$

$\frac{x^2}{36} - \frac{\frac{1}{4}x^2 - 9x + 81}{9} = 1$

$\frac{x^2}{36} - \frac{x^2 - 36x + 324}{36} = 1$

$x^2 - (x^2 - 36x + 324) = 36$

$36x - 324 = 36$

Question 5 continued

$36x = 360$

$x = 10$

$y = \frac{1}{2}(10) - 9$

$y = 5 - 9$

$y = -4$

$(10, -4) = Q$

$\frac{x^2}{36} - \frac{(-\frac{1}{2}x + 9)^2}{9} = 1$

$\frac{x^2}{36} - \frac{\frac{1}{4}x^2 - 9x + 81}{9} = 1$

$\frac{x^2}{36} - \frac{x^2 - 36x + 324}{36} = 1$

$x^2 - (x^2 - 36x + 324) = 36$

$36x - 324 = 36$

$36x = 360$

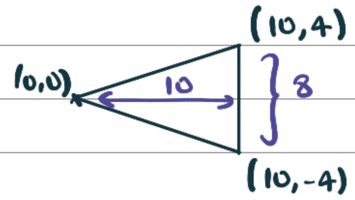
$x = 10$

$y = -\frac{1}{2}(10) + 9$

$y = -5 + 9$

$y = 4$

$(10, 4) = P$



$\frac{1}{2}(10)(8) = 40$

$\boxed{40 \text{ units}^2}$ //

(Total for Question 5 is 9 marks)

P 6 6 7 9 8 A 0 1 9 3 2

6. The Taylor series expansion of $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \ldots$$

Given that

$$y = (1 + \ln x)^2 \qquad x > 0$$

(a) show that $\dfrac{d^2y}{dx^2} = -\dfrac{2\ln x}{x^2}$ **(4)**

(b) Hence find $\dfrac{d^3y}{dx^3}$ **(2)**

(c) Determine the Taylor series expansion about $x = 1$ of

$$(1 + \ln x)^2$$

in ascending powers of $(x - 1)$, up to and including the term in $(x - 1)^3$

Give each coefficient in simplest form. **(3)**

(d) Use this series expansion to evaluate

$$\lim_{x \to 1} \frac{2x - 1 - (1 + \ln x)^2}{(x-1)^3}$$

explaining your reasoning clearly. **(3)**

a. $y = (1 + \ln x)^2$

use chain rule

$\dfrac{dy}{dx} = 2(1+\ln x)\left(\dfrac{1}{x}\right) = \dfrac{2}{x}(1+\ln x)$

remember to differentiate what is in the bracket too

$\dfrac{dy}{dx} = \underbrace{2x^{-1}}_{\alpha}\underbrace{(1+\ln x)}_{\beta}$ $\qquad \dfrac{d}{dx}(\alpha\beta) = \alpha'\beta + \alpha\beta'$

$\dfrac{d^2y}{dx^2} = -2x^{-2}(1+\ln x) + (2x^{-1})\left(\dfrac{1}{x}\right)$

$= -2x^{-2}(1+\ln x) + 2x^{-2}$

$= -2x^{-2} - 2x^{-2}\ln x + 2x^{-2}$

$= -2x^{-2}\ln x$

$\dfrac{d^2y}{dx^2} = -\dfrac{2\ln x}{x^2}$ //

P 6 6 7 9 8 A 0 2 0 3 2

Question 6 continued

b. $\frac{d^2y}{dx^2} = \underbrace{-2x^{-2}}_{u}\ \underbrace{\ln(x)}_{v}$ $\qquad \frac{d}{dx}(uv) = u'v + uv'$

$$\frac{d^3y}{dx^3} = 4x^{-3}\ln(x) - 2x^{-2}\left(\frac{1}{x}\right)$$

$$\frac{d^3y}{dx^3} = \frac{4\ln(x)}{x^3} - \frac{2}{x^3}$$

$$\frac{d^3y}{dx^3} = \frac{4\ln(x) - 2}{x^3} \quad //$$

c. @ $x=1$

$y = (1+\ln(1))^2 = 1$ $\qquad f(1) = 1$

$\left.\frac{dy}{dx}\right|_{x=1} 2(1+\ln(1))\left(\frac{1}{1}\right) = 2$ $\qquad f'(1) = 2$

$\left.\frac{d^2y}{dx^2}\right|_{x=1} \frac{-2\ln(1)}{(1)^2} = 0$ $\qquad f''(1) = 0$

$\left.\frac{d^3y}{dx^3}\right|_{x=1} \frac{4\ln(1) - 2}{(1)^3} = -2$ $\qquad f'''(1) = -2$

sub into taylor series eq^n above q

$$f(x) \approx 1 + 2(x-1) + \frac{(x-1)^2(0)}{2!} + \frac{(x-1)^3}{3!}(-2) + \ldots$$

$$f(x) \approx 1 + 2(x-1) - \frac{1}{3}(x-1)^3 + \ldots \quad //$$

d. $\lim_{x\to1} \frac{2x-1-\left(1+2(x-1) - \frac{(x-1)^3}{3} + \ldots\right)}{(x-1)^3}$

$\lim_{x\to1} \frac{\cancel{2x-1-1-2(x-1)} + \frac{(x-1)^3}{3} - \ldots}{(x-1)^3}$

Question 6 continued

we have not calculated these but for simplicity give variables.

$$\lim_{x\to1} \frac{\frac{(x-1)^3}{3} + \underbrace{\beta(x-1)^4 + \gamma(x-1)^5 + \dots}}{(x-1)^3}$$

$$\lim_{x\to1} \frac{1}{3} + \underbrace{\beta(x-1)^1 + \gamma(x-1)^2 + \dots}$$

will cancel out $(x-1)^3$ term to leave some remainder

as $x \to 1$, $(x-1) \to 0$

$\therefore$ only $1/3$ remains.

all other terms will cancel out no matter what variable is next to it.

$1/3$ //

7. With respect to a fixed origin O, the line l has equation

$$(\mathbf{r} - (12\mathbf{i} + 16\mathbf{j} - 8\mathbf{k})) \times (9\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}) = \mathbf{0}$$

The point A lies on l such that the direction cosines of $\overrightarrow{OA}$ with respect to the $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ axes are $\frac{3}{7}$, β and γ.

Determine the coordinates of the point A.

(7)

line eq^n in form: $(r - a) \times b = 0$

rewritten as $r = a + \lambda b$

$$r = \begin{pmatrix} 12 \\ 16 \\ -8 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ 6 \\ 2 \end{pmatrix} \equiv r = \begin{pmatrix} 12 + 9\lambda \\ 16 + 6t \\ -8 + 2t \end{pmatrix}$$

i - component

$$\cos\alpha = \frac{x}{|a|} = \frac{3}{7}$$

magnitude of i, j, k components

$$\frac{12 + 9t}{\sqrt{(12+9t)^2 + (16+6t)^2 + (-8+2t)^2}} = \frac{3}{7}$$

$$\left(\frac{12 + 9t}{\sqrt{(12+9t)^2 + (16+6t)^2 + (-8+2t)^2}}\right)^2 = \left(\frac{3}{7}\right)^2$$

square both sides to remove square root

$$\frac{(12+9t)^2}{(12+9t)^2 + (16+6t)^2 + (-8+2t)^2} = \frac{9}{49}$$

P 6 6 7 9 8 A 0 2 4 3 2

Question 7 continued

$$\frac{(12+9t)^2}{(12+9t)^2+(16+6t)^2+(-8+2t)^2} = \frac{9}{49}$$

$$\frac{81t^2+216t+144}{81t^2+216t+144+36t^2+192t+256+4t^2-32t+64} = \frac{9}{49}$$

$$\frac{81t^2+216t+144}{121t^2+376t+464} = \frac{9}{49}$$

$$49(81t^2+216t+144) = 9(121t^2+376t+464)$$

$$3969t^2+10584t+7056 = 1089t^2+3384t+4176$$

$$2880t^2+7200t+2880=0$$

$\downarrow \div 1440$

$$2t^2+5t+2=0$$

$$(2t+1)(t+2)=0$$

$$t=-\tfrac{1}{2} \text{ or } t=-2$$

sub in $t=-\frac{1}{2}$ into gen eq^n of line

@ $t=-\frac{1}{2}$

$$r=\begin{pmatrix}12+9\left(-\frac{1}{2}\right)\\16+6\left(-\frac{1}{2}\right)\\-8+2\left(-\frac{1}{2}\right)\end{pmatrix}=\begin{pmatrix}\frac{15}{2}\\13\\-9\end{pmatrix}$$

@ $t=-2$

$$r=\begin{pmatrix}12+9(-2)\\16+6(-2)\\-8+2(-2)\end{pmatrix}=\begin{pmatrix}-6\\4\\-12\end{pmatrix}$$

direction cosine is +ve. so i-component must be positive.

$\therefore t=-\frac{1}{2}$

Question 7 continued

can also find $\cos\theta_x$ for both vectors and see if $\cos\theta_x = \frac{3}{7}$

for $t = -1/2$

$$\cos\theta_x = \frac{15/2}{\sqrt{\left(\frac{15}{2}\right)^2 + (13)^2 + (-9)^2}} = \frac{3}{7}$$

for $t = -2$

$$\cos\theta_x = \frac{-6}{\sqrt{(-6)^2 + (4)^2 + (-12)^2}} = -\frac{3}{7}$$

-ve direction
so $t \neq -2$

$A: \left(\frac{15}{2}, 13, -9\right)$

P 6 6 7 9 8 A 0 2 6 3 2

8. A community is concerned about the rising level of pollutant in its local pond and applies a chemical treatment to stop the increase of pollutant.

The concentration, x parts per million (ppm), of the pollutant in the pond water t days after the chemical treatment was applied, is modelled by the differential equation

$$\frac{dx}{dt} = \frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t \qquad \text{(I)}$$

When the chemical treatment was applied the concentration of pollutant was 3 ppm.

(a) Use the iteration formula

$$\left(\frac{dx}{dt}\right)_n \approx \frac{x_{n+1}-x_n}{h} \qquad \left(\frac{dy}{dx}\right)_n \approx \frac{(y_{n+1}-y_n)}{h}$$

once to estimate the concentration of the pollutant in the pond water 6 hours after the chemical treatment was applied.

(4)

(b) Show that the transformation $u = x^3$ transforms the differential equation (I) into the differential equation

$$\frac{du}{dt} + u\tanh t = 1 + \frac{3}{\cosh t} \qquad \text{(II)}$$

(3)

(c) Determine the general solution of equation (II)

(4)

(d) Hence find an equation for the concentration of pollutant in the pond water t days after the chemical treatment was applied.

(3)

(e) Find the percentage error of the estimate found in part (a) compared to the value predicted by the model, stating if it is an overestimate or an underestimate.

(3)

a. @ $t_0 = 0 \quad x_0 = 3$

6 hrs in days = $1/4$

$\therefore h = 0.25$

$$\left.\frac{dx}{dt}\right|_0 \quad \frac{3+\cosh(0)}{3(3)^2\cosh(0)} - \frac{1}{3}(3)\tanh(0) = \frac{4}{27}$$

$t_0 = 0, \; x_0 = 3$

Question 8 continued

$$x_1 \approx x_0 + h\left(\frac{dx}{dt}\right)_0$$

$$x_1 \approx 3 + \frac{1}{4}\left(\frac{4}{27}\right)$$

$$x_1 \approx \frac{82}{27}$$

$\frac{82}{27}$ ppm //

b. $u = x^3$

$$\frac{du}{dx} = 3x^2$$

flip

$$\frac{dx}{dt} = \frac{dx}{du} \times \frac{du}{dt}$$

$$\boxed{\frac{dx}{dt} = \frac{1}{3x^2}\frac{du}{dt}}$$

$$\frac{dx}{dt} = \frac{3 + \cosh t}{3x^2 \cosh t} - \frac{1}{3}x \tanh t$$

$$\frac{dx}{dt} = \frac{\cancel{3}}{\cancel{3}x^2 \cosh t} + \frac{\cancel{\cosh t}}{3x^2 \cancel{\cosh t}} - \frac{1}{3}x \tanh t$$

$$\frac{dx}{dt} = \frac{1}{x^2 \cosh t} + \frac{1}{3x^2} - \frac{1}{3}x \tanh t$$

replace w/ red box

$$\frac{1}{3x^2}\frac{du}{dt} = \frac{1}{x^2 \cosh t} + \frac{1}{3x^2} - \frac{1}{3}x \tanh t$$

Question 8 continued

$\frac{du}{dt} = \frac{3}{\cosh t} + 1 - x^3\tanh t$ (multiply both sides by $3x^2$)

$\frac{du}{dt} + x^3\tanh t = 1 + \frac{3}{\cosh t}$

$u = x^3$ put back in terms of u

$\frac{du}{dt} + u\tanh t = 1 + \frac{3}{\cosh t}$ //

c. use integrating factor: $e^{\int \tanh t \, dt} = \cancel{e}^{\cancel{\ln}(\cosh t)} = \cosh t$

$\frac{du}{dt}\cosh t + u\left(\frac{\sinh t}{\cancel{\cosh t}}\right)\cancel{\cosh t} = \cosh t\left(1 + \frac{3}{\cosh t}\right)$

tanh t

$\cosh t \frac{du}{dt} + u\sinh t = \cosh t + 3$

$\Updownarrow$ equivalent

$\frac{d}{dt}(u\cosh t) = \cosh t + 3$

$u\cosh t = \int(\cosh t + 3)\, dt$

$u\cosh t = \sinh t + 3t + c$

convert back to x

$x^3\cosh t = \sinh t + 3t + c$

@ $t = 0$, $x = 3$

$27(1) = (0) + 3(0) + c$

$c = 27$

$x^3\cosh t = \sinh t + 3t + 27$ //

Question 8 continued

a. $x = \left(\tanh t + \frac{3t+27}{\cosh t}\right)^{1/3}$

@ $t = 1/4$ concentration was applied.

$x = \left(\tanh(1/4) + \frac{3(1/4)+27}{\cosh(1/4)}\right)^{1/3}$

$x = 3.0055...$ //

e.

% error :

$\frac{\text{true value} - \text{estimated value}}{\text{estimated value}} \times 100$

$\frac{82/27 - 3.0055}{3.0055} \times 100$

$\approx 1.05\ \%$

$\therefore$ overestimates (a) by $\sim 1.05\ \%$ //